Our study of the precession of the perihelion of Mercury has illustrated several useful techniques. One is the use of extrapolation to deal with situations in which the effects of interest are too small to conveniently estimate directly with a numerical approach. Of course, you must have some confidence in how the extrapolation should be made. In the present case we were able to show that a linear extrapolation as a function of α is appropriate. The second useful technique we have introduced is the method of least squares. We will use it again in later chapters.

EXERCISES

4.10. Calculate the precession of the perihelion of Mercury, following the approach described in this section.

***4.11.** Investigate how the precession of the perihelion of a planet's orbit due to general relativity varies as a function of the eccentricity of the orbit. Study the precession of different elliptical orbits with different eccentricities, but with the same value of the perihelion. Let the perihelion have the same value as for Mercury, so that you can compare it with the results shown in this section.

potential small project

4.4 THE THREE-BODY PROBLEM AND THE EFFECT OF JUPITER ON EARTH

To this point all of our planetary simulations have involved two-body solar systems. It is now time to consider some of the things that can happen when there are three or more objects in the solar system. The problem of two objects interacting through the inverse-square law (4.1) can be solved exactly (as we have already mentioned), leading to Kepler's laws. However, if we add just one more planet to give what is known as the *three-body problem*, an analytic theory becomes *much* more difficult. In fact, there are very few exact results in this case, even though it has been studied extensively for several centuries. Indeed, the three-body, or more generally the n-body problem, is *the* problem of celestial mechanics.

In this section we consider one of the simplest three-body problems, the Sun and two planets, which we will take to be Earth and Jupiter. We know that without Jupiter, Earth's orbit is stable and unchanging with time.[16] Our objective is to observe how much effect the gravitational force from Jupiter has on Earth's motion. We consider Jupiter since, at about 0.1% of the solar mass, it is by far the largest planet in the solar system.

To carry out this simulation we must modify our planetary-motion program to include two planets and the gravitational force between them. The magnitude of the force between Jupiter and Earth is given by our now familiar inverse-square law, with the Sun replaced by Jupiter

$$F_{E,J} = \frac{G\,M_J\,M_E}{r_{EJ}^2}\,, \tag{4.17}$$

where M_J is the mass of Jupiter and r_{EJ} the distance between Earth and Jupiter. We assume that the orbits of the two planets are coplanar, and we take this to be

[16]Ignoring general relativity, whose effect is much smaller for Earth than it is for Mercury.

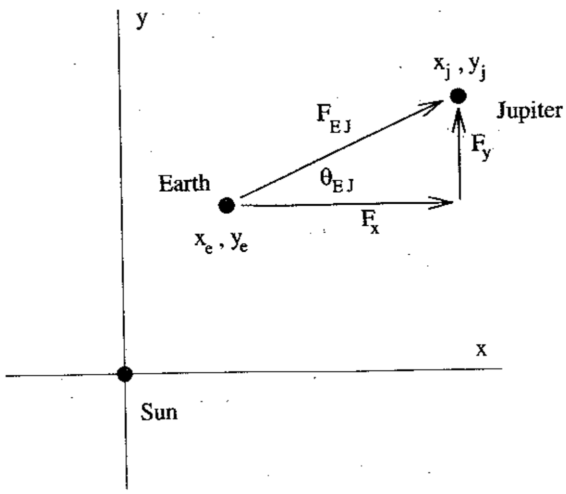

FIGURE 4.11: Components of the gravitational force due to Jupiter, located at x_j, y_j, with Earth at x_e, y_e. The Sun is at the origin.

the xy-plane. Writing $F_{E,J}$ in terms of components yields (see Figure 4.11)

$$F_{EJ,x} = -\frac{G\,M_J\,M_E}{r_{EJ}^2}\cos\theta_{EJ} = -\frac{G\,M_J\,M_E\,(x_e - x_j)}{r_{EJ}^3}, \tag{4.18}$$

for the x component of the force, with a corresponding result for the y component. Here x_e and x_j are the coordinates of Earth and Jupiter (the Sun remains at the origin), and θ_{EJ} is the angle defined in Figure 4.11. The total force on Earth in the x direction will be the sum of the forces of gravity from the Sun (4.3) and Jupiter (4.18), yielding the equation of motion for the x component of Earth's velocity, $v_{x,e}$

$$\frac{dv_{x,e}}{dt} = -\frac{G\,M_S\,x_e}{r^3} - \frac{G\,M_J\,(x_e - x_j)}{r_{EJ}^3}, \tag{4.19}$$

where r is again the distance from Earth to the Sun. We can convert this and the corresponding result for $v_{y,e}$, into difference equations, just as we did in (4.7). Since M_S is much greater than M_J or M_E, we may treat the Sun to be stationary at the origin and just calculate the positions of Jupiter and Earth. Thus, the only thing left to do is calculate GM_J in the appropriate units. Here it is simplest to use the result $GM_J = GM_S(M_J/M_S) = 4\pi^2(M_J/M_S)$, with M_J and M_S given in Table 4.1. A program to calculate the orbits of two planets in this approximation is a straightforward extension of the one we sketched for the two-body solar system previously in Example 4.1. We now need to update the positions and velocities of both planets at each step through the main loop. A sketch of such a calculation is given in Example 4.2.

EXAMPLE 4.2 Subroutine `jupiter-earth` for a two-planet solar system

- At each time step i, calculate the positions (x_e, y_e) (Earth) and (x_j, y_j) (Jupiter) as well as the velocities $(v_{e,x}, v_{e,y})$ and $(v_{j,x}, v_{j,y})$ for the time $i+1$ using the Euler-Cromer method.
  - ▷ Calculate the distances among Earth, Jupiter, and the Sun:
    $r_e(i) = \sqrt{x_e(i)^2 + y_e(i)^2}$,
    $r_j(i) = \sqrt{x_j(i)^2 + y_j(i)^2}$,
    $r_{EJ} = \sqrt{[x_e(i) - x_j(i)]^2 + [y_e(i) - y_j(i)]^2}$.
  - ▷ Compute the new velocity of Earth
    $v_{e,x}(i+1) = v_{e,x}(i) - \frac{4\pi^2 x_e(i)}{r_e(i)^3}\Delta t - \frac{4\pi^2 (M_J/M_S)[x_e(i)-x_j(i)]}{r_{EJ}(i)^3}\Delta t$,
    and similarly for the y-component, $v_{e,y}(i+1)$.
  - ▷ Compute the new velocity of Jupiter
    $v_{j,x}(i+1) = v_{j,x}(i) - \frac{4\pi^2 x_j(i)}{r_j(i)^3}\Delta t - \frac{4\pi^2 (M_E/M_S)[x_j(i)-x_e(i)]}{r_{EJ}(i)^3}\Delta t$,
    and similarly for the y-component, $v_{j,y}(i+1)$.
  - ▷ Use the Euler-Cromer method to calculate the new positions of Earth and Jupiter:
    $x_e(i+1) = x_e(i) + v_{e,x}(i+1)\Delta t$ // $y_e(i+1) = y_e(i) + v_{e,y}(i+1)\Delta t$ //
    and similarly for Jupiter.
  - ▷ Record or plot the new positions as they become available.
  - ▷ Repeat for desired number of time steps.

A note about program efficiency should also be made here. You may have noticed that the Euler-Cromer equations used above contain factors such as $4\pi^2 M_J/M_S$. The way we have written the code, these factors may be recomputed each time through the loop. However, we could have arranged to calculate them just once at the beginning of the program, thereby saving some time during each iteration. As we have noted in previous chapters, it is usually our policy to sacrifice speed for clarity. You might argue that with the proper use of comment statements the reduction of clarity can be made very small. While this is correct, we would argue in response that the increase in program speed is probably also very small. Indeed, most modern compilers and interpreters would probably recognize that our factors of $4\pi^2 M_J/M_S$ do not have to be recomputed each time and would store them accordingly. Hence, we would have both our clarity and efficiency. In any case, if you do insist on expending extra effort to make a program execute as fast as possible, we have two recommendations. First, don't spend much time at this until *after* the program is working. Second, use a profiling tool or other means to determine which part(s) of the program is limiting the overall speed. Chances are that only one routine or loop will need to be tuned, so you might as well concentrate your efforts on it.

Some results from this simulation are shown in Figure 4.12. Using parameters appropriate for Earth and Jupiter, we find that both of the planets follow stable

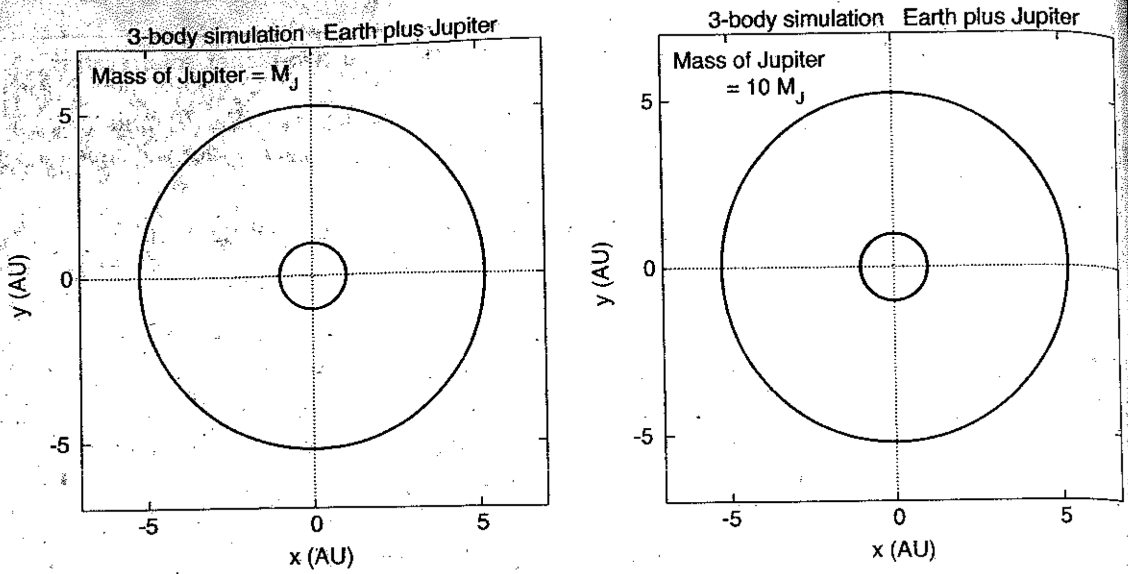

FIGURE 4.12: Simulation of a solar system with two planets, Earth and Jupiter. Left: Jupiter has its true mass; right: the mass of Jupiter has been set to 10 times its true mass.

circular orbits. Thus, Jupiter has a negligible effect on Earth (at least on this scale). This should not be terribly surprising; we know that Earth has been orbiting the Sun for several billion years, so the orbit must be fairly stable! We can also use our simulation to calculate what would happen if the mass of Jupiter were somehow increased. Giving Jupiter a mass of $10M_J$, that is, 10 times its actual value, has no discernible effect on Earth (at least on the scale of this figure).

It is tempting to see what would happen if the mass of Jupiter were increased to $100M_J$ or even to $1000M_J$, using the same program. However, since $1000M_J$ is about equal to the mass of the Sun, the perturbation by Jupiter *on* the Sun would then be significant and would have to be taken into account. If we ignore this detail(!), and simply use $1000M_J$ as the mass of the second planet in routine `jupiter-earth`, we find the interesting trajectory shown on the left in Fig. 4.13. We see that Earth's orbit becomes completely unstable, as it is eventually ejected from the solar system! To investigate this problem more carefully, we need to perform a true three-body calculation, in which the motion of *all three* bodies – the Sun, Jupiter, and Earth – are computed. The results of such a simulation are very sensitive to the initial conditions, and often quite dramatic. For example, it is possible for Earth to switch back and forth between approximate orbits centered on Jupiter and the Sun. An example of such a true three-body simulation is shown on the right side of Figure 4.13. We will let you explore this problem further in the exercises.

The conclusion from this simulation is that Jupiter is (fortunately) too small to have a major influence on Earth. However, Jupiter is much closer to Mars,

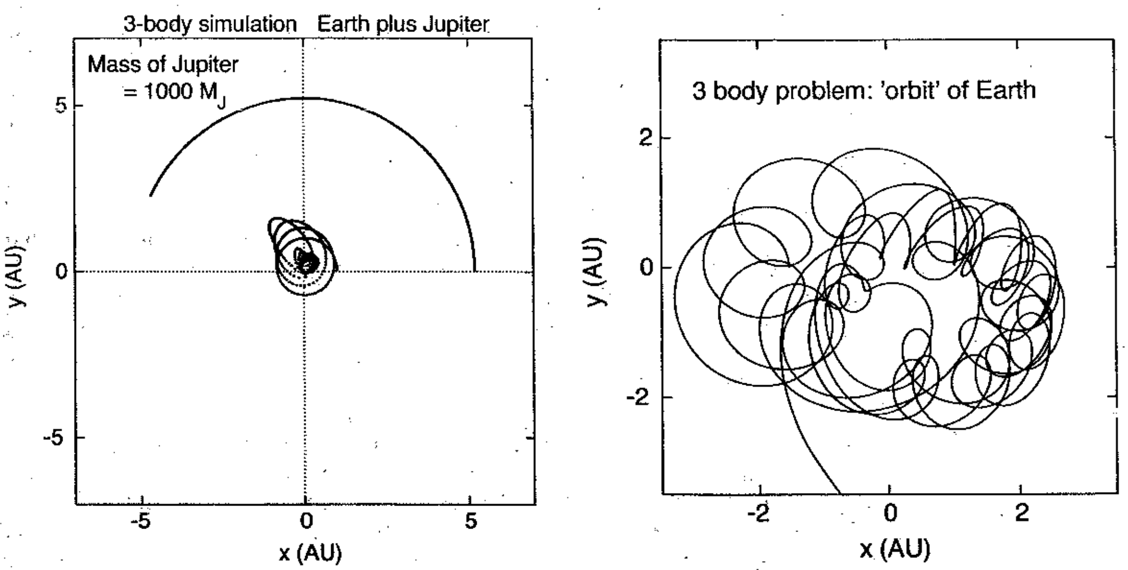

FIGURE 4.13: Simulation of a solar system with two planets, Earth and Jupiter. Left: the mass of Jupiter has been set to 1000 times its true mass, and we have used the routine `jupiter-earth`, which does *not* take into account the motion of the Sun. Here we stopped the simulation before Jupiter had completed even half an orbit, as the motion of Earth was unstable. Right: Typical results for a true 3-body simulation, in which the motions of Earth, Jupiter, and the Sun were all computed. Here we show only the motion of the Earth. The origin is now the center of mass of the 3-body system.

so there will be a larger effect in that case. We will leave it to the exercises to investigate this issue. In the next section we will consider the effect of Jupiter on the asteroids that orbit the Sun in the region between Mars and Jupiter.

EXERCISES

4.12. Investigate the effect of Jupiter on Mars.

***4.13.** Explore the orbits of a planet in a double-star system. Write a program that computes the motion of both stars along with that of the planet, including the gravitational forces of both stars on the planet and on each other. Explore the nature of the planetary orbits, and try to discover stable, repeating ones. *Hint:* First consider the case of two stars of equal mass, and pay special attention to planetary orbits that are especially symmetric with respect to the orbits of the stars.

4.14. Simulate the orbits of Earth and Moon in the solar system by writing a program that accurately tracks the motions of both as they move about the Sun. Be careful about (1) the different time scales present in this problem, and (2) the correct initial velocities (i.e., set the initial velocity of Moon taking into account the motion of Earth around which it orbits).

***4.15.** In our discussion of the precession of the perihelion of Mercury we mentioned that the other planets cause most of the observed precession. As the largest planet, Jupiter is responsible for most of this. Calculate the precession of the perihelion of Mercury due to Jupiter. We suggest that you perform the calculation by giving Jupiter a mass that is much larger than its true value, and then extrapolate to obtain the final result.

***4.16.** Carry out a true three-body simulation in which the motions of Earth, Jupiter, and the Sun are all calculated. Since all three bodies are now in motion, it is useful to take the center of mass of the three-body system as the origin, rather than the position of Sun. We also suggest that you give Sun an initial velocity which makes the total momentum of the system exactly zero (so that the center of mass will remain fixed). Study the motion of Earth with different initial conditions. Also, try increasing the mass of Jupiter to 10, 100, and 1000 times its true mass.

4.5 RESONANCES IN THE SOLAR SYSTEM: KIRKWOOD GAPS AND PLANETARY RINGS

Table 4.1 gives the distances between the nine planets and the Sun. It is interesting to note that these distances grow in a seemingly regular manner, with the spacing between adjacent planets increasing as we go outward from the Sun. In fact, this pattern can be described by what is known as the Titus-Bode formula (see Peterson [1993]). According to that formula, the distances from the planets to the Sun are closely related to the sequence of integers

$$0, 3, 6, 12, 24, \ldots \tag{4.20}$$

which is obtained by starting with the integers 0 and 3 and letting each succeeding term be just twice the term before. To derive the distance from a planet to the Sun, the corresponding term in this sequence is added to 4 and the result divided by 10 (in AU). In this formula, the first planet is Mercury, the second Venus, etc.