The `ThreeBodyApp` class in Listing 5.10 is the target class for the three-body program. The `doStep` method merely increments the model's differential equations solver and repaints the view.

Listing 5.10 A program that displays the trajectories of three bodies interacting via gravitational forces.

```
package org.opensourcephysics.sip.ch05;
import org.opensourcephysics.controls.*;
import org.opensourcephysics.frames.*;

public class ThreeBodyApp extends AbstractSimulation {
    PlotFrame frame = new PlotFrame("x", "y", "Three-Body Orbits");
    ThreeBody trajectory = new ThreeBody();

    public ThreeBodyApp() {
        frame.addDrawable(trajectory);
        frame.setSquareAspect(true);
        frame.setSize(450, 450);
    }

    public void initialize() {
        trajectory.odeSolver.setStepSize(control.getDouble("dt"));
        trajectory.initialize(ThreeBodyInitialConditions.MONTGOMERY);
        frame.setPreferredMinMax(-1.5, 1.5, -1.5, 1.5);
    }

    public void reset() {
        control.setValue("dt", 0.1);
        enableStepsPerDisplay(true);
        initialize();
    }

    protected void doStep() {
        trajectory.doStep();
        frame.setMessage("t="+decimalFormat.format(trajectory.state[4]));
    }

    public static void main(String[] args) {
        SimulationControl.createApp(new ThreeBodyApp());
    }
}
```

Problem 5.16 Stability of solutions to the three-body problem
Examine the stability of the three solutions to the three-body problem by slightly varying the initial velocity of one of the masses. Before passing your new initial state to `trajectory.initialize`, calculate the center of mass velocity and subtract this velocity from every object. Show that any instability is due to the physics and not to the numerical differential equation solver. Which of the three analytic solutions is stable? Check conservation of the total energy and angular momentum. ∎

5.12 ∎ PROJECTS

Project 5.17 Effect of a "solar wind"

(a) Assume that a satellite is affected not only by the Earth's gravitational force, but also by a weak uniform "solar wind" of magnitude W acting in the horizontal direction. The equations of motion can be written as

$$\frac{d^2x}{dt^2} = -\frac{GMx}{r^3} + W \qquad (5.32a)$$

$$\frac{d^2y}{dt^2} = -\frac{GMy}{r^3}. \qquad (5.32b)$$

Choose initial conditions so that a circular orbit would be obtained for $W = 0$. Then choose a value of W whose magnitude is about 3% of the acceleration due to the gravitational field and compute the orbit. How does the orbit change?

(b) Determine the change in the velocity space orbit when the solar wind (5.32) is applied. How does the total angular momentum and energy change? Explain in simple terms the previously observed change in the position space orbit. See Luehrmann for further discussion of this problem. ∎

Project 5.18 Resonances and the asteroid belt

(a) A histogram of the number of asteroids versus their distance from the Sun shows some distinct gaps. These gaps, called the *Kirkwood gaps*, are due to resonance effects. That is, if asteroids were in these gaps, their periods would be simple fractions of the period of Jupiter. Modify class `Planet2` so that planet two has the mass of Jupiter by setting `GM1 = 0.001*GM`. Because the asteroid masses are very small compared to that of Jupiter, the gravitational force on Jupiter due to the asteroids can be neglected. The initial conditions listed in `Planet2` are approximately correct for Jupiter. The initial conditions for the asteroid (planet one in `Planet2`) correspond to the 1/3 resonance (the period of the asteroid is one third that of Jupiter). Run the program with these changes and describe the orbit of the asteroid.

(b) Use Kepler's third law, T^2/a^3 = constant, to determine the values of a, the asteroid's semimajor axis, such that the ratio of its period of revolution about the Sun to that of Jupiter is 1/2, 3/7, 2/5, and 2/3. Set the initial value of `x(1)` equal to a for each of these ratios and choose the initial value of `vy(1)` so that the asteroid would have a circular orbit if Jupiter was not present. Describe the orbits that you obtain.

(c) It is instructive to plot a as a function of time. However, because it is not straightforward to measure a directly in the simulation, it is more convenient to plot the quantity $-2GMm/E$, where E is the total energy of the asteroid and m is the mass of the asteroid. Because E is proportional to m, the quantity $-2GMm/E$ is independent of m. If the interaction of the asteroid with Jupiter is ignored, it can be shown that $a = -2GMm/E$, where E is the asteroid kinetic energy plus the asteroid-Sun potential energy. Derive this result for circular orbits. Plot the quantity $-2GMm/E$ versus time for about thirty revolutions for the initial conditions in Problem 5.18b.

(d) Compute the time dependence of $-2GMm/E$ for asteroid orbits whose initial position $x(1)$ ranges from 2.0 to 5.0 in steps of 0.2. Choose the initial values of $vy(1)$ so that circular orbits would be obtained in the absence of Jupiter. Are there any values of $x(1)$ for which the time dependence of a is unusual?

(e) Make a histogram of the number of asteroids versus the value of $-2GMm/E$ at $t = 2000$. (You can use the HistogramFrame class described on page 206 if you wish.) Assume that the initial value of $x(1)$ ranges from 2.0 to 5.0 in steps of 0.02 and choose the initial values of $vy(1)$ as before. Use a histogram bin width of 0.1. If you have time, repeat for $t = 5000$ and compare the histogram with your previous results. Is there any evidence for Kirkwood gaps? A resonance occurs when the periods of the asteroid and Jupiter are related by simple fractions. We expect the number of asteroids with values of a corresponding to resonances to be small.

(f) Repeat part (e) with initial velocities that vary from their values for a circular orbit by 1, 3, and 5%. ∎

Project 5.19 The classical helium atom

The classical helium atom is a relatively simple example of a three-body problem and is similar to the gravitational three-body problem of a heavy sun and two light planets. The important difference is that the two electrons repel one another, unlike the planetary case where the intraplanetary interaction is attractive. If we ignore the small motion of the heavy nucleus, the equations of motion for the two electrons can be written as

$$\mathbf{a}_1 = -2\frac{\mathbf{r}_1}{r_1^3} + \frac{\mathbf{r}_1 - \mathbf{r}_2}{r_{12}^3} \tag{5.33a}$$

$$\mathbf{a}_2 = -2\frac{\mathbf{r}_2}{r_2^3} + \frac{\mathbf{r}_2 - \mathbf{r}_1}{r_{12}^3}, \tag{5.33b}$$

where $\mathbf{r}_1$ and $\mathbf{r}_2$ are measured from the fixed nucleus at the origin, and r_{12} is the distance between the two electrons. We have chosen units such that the mass and charge of the electron are both unity. The charge of the helium nucleus is two in these units. Because the electrons are sometimes very close to the nucleus, their acceleration can become very large, and a very small time step Δt is required. It is not efficient to use the same small time step throughout the simulation, and instead a variable time step or an *adaptive* step size algorithm is suggested. An adaptive step size algorithm can be used with any standard numerical algorithm for solving differential equations. The RK45 algorithm described in Appendix 3A is adaptive and is a good all-around choice for these types of problems.

(a) For simplicity, we restrict our atom to two dimensions. Modify Planet2 to simulate the classical helium atom. Choose units such that the electron mass is one and the other constants are absorbed into the unit of charge so that the force between two electrons is

$$|F| = \frac{1}{r^2}. \tag{5.34}$$

Choose the initial value of the time step to be $\Delta t = 0.001$. Some of the possible orbits are similar to those we have seen in our mini-solar system. For example, try the initial condition $\mathbf{r}_1 = (2, 0)$, $\mathbf{r}_2 = (-1, 0)$, $\mathbf{v}_1 = (0, 0.95)$, and $\mathbf{v}_2 = (0, -1)$.
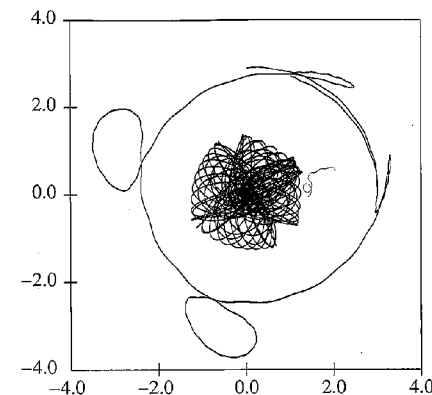
Figure 5.8 Orbits of the two electrons in the classical helium atom with the initial condition $\mathbf{r}_1 = (3, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$ (see Project 5.19c).

(b) Most initial conditions result in unstable orbits in which one electron eventually leaves the atom (autoionization). The initial condition $\mathbf{r}_1 = (1.4, 0)$, $\mathbf{r}_2 = (-1, 0)$, $\mathbf{v}_1 = (0, 0.86)$, and $\mathbf{v}_2 = (0, -1)$ gives "braiding" orbits. Make small changes in this initial condition to observe autoionization.

(c) The classical helium atom is capable of very complex orbits (see Figure 5.8). Investigate the motion for the initial condition $\mathbf{r}_1 = (3, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$. Does the motion conserve the total angular momentum? Also try $\mathbf{r}_1 = (2.5, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{v}_1 = (0, 0.4)$, and $\mathbf{v}_2 = (0, -1)$.

(d) Choose the initial condition $\mathbf{r}_1 = (2, 0)$, $\mathbf{r}_2 = (-1, 0)$, and $\mathbf{v}_2 = (0, -1)$. Then vary the initial value of $\mathbf{v}_1$ from $(0.6, 0)$ to $(1.3, 0)$ in steps of $\Delta v = 0.02$. For each set of initial conditions, calculate the time it takes for autoionization. Assume that ionization occurs when either electron exceeds a distance of six from the nucleus. Run each simulation for a maximum time of 2000. Plot the ionization time versus v_{1x}. Repeat for a smaller interval of Δv centered about one of the longer ionization times. These calculations require much computer resources. Do the two plots look similar? If so, such behavior is called "self-similar" and is characteristic of chaotic systems and the geometry of fractals (see Chapters 6 and 13). More discussion on the nature of the orbits can be found in Yamamoto and Kaneko. ∎

REFERENCES AND SUGGESTIONS FOR FURTHER READING

Harold Abelson, Andrea diSessa, and Lee Rudolph, "Velocity space and the geometry of planetary orbits," Am. J. Phys. **43**, 579–589 (1975). See also Andrea diSessa, "Orbit: a mini-environment for exploring orbital mechanics," O. Lecarme and R. Lewis, editors, *Computers in Education*, 359, North-Holland (1975). Detailed geometrical rather than calculus-based arguments on the origin of closed orbits for inverse-square forces are presented. Sections 5.7 and 5.8 are based on these papers.