ZCT 104/3E Modern Physics
Semester II, Sessi 2006/07
Open Book Quiz I (22 Dec 2007)
Duration: 30 min

Name:

Matrics No:

INSTRUCTION: Answer both following questions. Note that question 2 is printed at the opposite page. Each question carries 10 marks.

1. Derive time dilation effect $\Delta\tau = \Delta t/\gamma$ by using the Lorentz transformation formula, where $\Delta\tau$ is the proper time interval, Δt the improper time interval, and γ is the Lorentz factor, $\gamma = \dfrac{1}{\sqrt{1-\frac{v^2}{c^2}}}$.

ANS

Lorentz transformation: $x' = \gamma(x - vt), t' = \gamma\left(t - \frac{v}{c^2}x\right)$.

Say two events are happening at the same point one after another in the primed frame, which is moving with a constant velocity v with respect to an unprimed frame. The proper time between these two events is $\Delta\tau = t_2' - t_1'$. By definition, these two events are happening in the same point in the primed frame, hence $x_2' - x_1' = 0$.

The temporal interval of these two events as observed in the unprimed frame, $\Delta t = t_2 - t_1$, according to LT, could be related to $\Delta\tau = t_2' - t_1'$ via LT as

$$\Delta\tau = t_2' - t_1' = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) - \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma(t_2 - t_1) - \frac{\gamma v}{c^2}(x_2 - x_1) = \gamma\Delta t - \frac{\gamma v}{c^2}(x_2 - x_1),$$

where x_2 and x_1 are the event sites as observed in the unprimed frame. Within the temporal interval of Δt, the primed frame has moved through a distance of $v\Delta t$ (as observed by an observer in the unprimed frame), which is equal to the displacement of the two event sites from the unprimed frame point of view: $(x_2 - x_1) = v\Delta t$.

Hence, $\Delta\tau = \gamma\Delta t - \frac{\gamma v}{c^2}(v\Delta t) = \gamma\Delta t - \frac{\gamma v^2}{c^2}\Delta t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta t\left(1 - \frac{v^2}{c^2}\right) = \Delta t\sqrt{1-\frac{v^2}{c^2}} = \Delta t/\gamma$.

Hence we recover time dilation formula: $\Delta\tau = \Delta t/\gamma$

2. An object of rest mass M decays into two daughter objects of rest mass m_1 and m_2 respectively. Calculate the kinetic energy for each of the daughter masses respectively in terms of M, m_1 and m_2.

ANS

By conservation of energy: total energy before decay = total energy after decay:

$$E = E_1 + E_2,$$

where

$E = Mc^2$ $E_1 = K_1 + m_1c^2$, $E_2 = K_2 + m_2c^2$

$\Rightarrow Mc^2 = (K_1 + K_2) + (m_1 + m_2)c^2$ Eq(1)

For daughter mass 1:

$E_1{}^2 = p_1{}^2 c^2 + m_1{}^2 c^4$. Eq. (2)

Likewise, for daughter mass 2:

$E_2{}^2 = p_2{}^2 c^2 + m_2{}^2 c^4$ Eq. (3)

Due to conservation of momentum: $|\vec{p}_2| \equiv p_2 = |\vec{p}_1| \equiv p_1 \equiv p$

Eq. (3) - Eq. (2):

$E_2{}^2 - E_1{}^2 = (m_2{}^2 - m_1{}^2)c^4$

$\Rightarrow (E_2 - E_1)(E_2 + E_1) = (m_2{}^2 - m_1{}^2)c^4$ Eq. (4)

Substitute $E_1 = K_1 + m_1c^2$, $E_2 = K_2 + m_2c^2$ and $E_1 + E_2 = E = Mc^2$ into Eq. (4),

$\Rightarrow [(K_2 - K_1) + (m_2 - m_1)c^2](Mc^2) = (m_2{}^2 - m_1{}^2)c^4$

$\Rightarrow (K_2 - K_1) = \left(\frac{m_2{}^2 - m_1{}^2}{M}\right)c^2 - (m_2 - m_1)c^2$ Eq. (5)

We can then solve for K_1 and K_2 from Eq. (5) and Eq. (1):

Eq. (5) + Eq. (6):

$$\Rightarrow 2K_2 = Mc^2 - (m_1 + m_2)c^2 + \left(\frac{m_2{}^2 - m_1{}^2}{M}\right)c^2 - (m_2 - m_1)c^2$$

$$\Rightarrow K_2 = \frac{1}{2}\left[M - m_2\left(2 - \frac{m_2}{M}\right) - \frac{m_1{}^2}{M}\right]c^2$$

Eq. (5) - Eq. (6):

$$\Rightarrow 2K_1 = Mc^2 - (m_1 + m_2)c^2 - \left(\frac{m_2{}^2 - m_1{}^2}{M}\right)c^2 + (m_2 - m_1)c^2$$

$$\Rightarrow K_1 = \frac{1}{2}\left[M - 2m_1 - \left(\frac{m_2{}^2 - m_1{}^2}{M}\right)\right]c^2 = \frac{1}{2}\left[M - m_1\left(2 - \frac{m_1{}^2}{M}\right) - \left(\frac{m_2{}^2}{M}\right)\right]c^2$$